arXiv:math/0404351v1 [math.AG] 20 Apr 2004

COHOMOLOGY OF THE MODULI SPACE OF HECKE CYCLES

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Abstract. Let X be a smooth projective curve of genus $g \geq 3$ and let M_0 be the moduli space of semistable bundles over X of rank 2 with trivial determinant. Three different desingularizations of M_0 have been constructed by Seshadri [Ses77], Narasimhan-Ramanan [NR78], and Kirwan [Kir86b]. In this paper, we construct a birational morphism from Kirwan's desingularization to Narasimhan-Ramanan's, and prove that the Narasimhan-Ramanan's desingularization (called the moduli space of Hecke cycles) is the intermediate variety between Kirwan's and Seshadri's as was conjectured recently in [KL04]. As a by-product, we compute the cohomology of the moduli space of Hecke cycles.

1. Introduction

Let X be a smooth projective curve of genus $g \geq 3$ over the complex number field. Let M_0 be the moduli space of semistable bundles over X of rank 2 with trivial determinant. Then M_0 is a singular normal projective variety of dimension $3g-3$. Its singular locus is the Kummer variety $\mathfrak{K}$ which consists of the S-equivalence classes of strictly semistable bundles $E = L \oplus L^{-1}$ for $L \in Pic^0(X)$.

There are three different constructions to desingularize M_0:

(1) Seshadri's desingularization $\mathbf{S}$ ([Ses77]),

(2) Narasimhan-Ramanan's desingularization $\mathbf{N}$ ([NR78]), called the *moduli space of Hecke cycles*, and

(3) Kirwan's desingularization $\mathbf{K}$ ([Kir86b]).

The first two desinglarizations $\mathbf{S}$ and $\mathbf{N}$ come from certain moduli problems, while $\mathbf{K}$ is obtained as a result of more general construction of a partial desingularization of a GIT quotient, which was studied by F. Kirwan in [Kir85].

Recently Y.-H. Kiem and J. Li in [KL04] constructed a morphism $f : \mathbf{K} \to \mathbf{S}$ and described it explicitly as a composition of two blow-downs:

$$f : \qquad \mathbf{K} \xrightarrow{f_\sigma} \mathbf{K}_\sigma \xrightarrow{f_\epsilon} \mathbf{K}_\epsilon \ (\cong \mathbf{S}).$$

Also they conjectured that the intermediate variety $\mathbf{K}_\sigma$ is isomorphic to the moduli space of Hecke cycles $\mathbf{N}$ ([KL04], Conjecture 5.7). In this paper, we give a proof of this conjecture and compute the cohomology of $\mathbf{N}$ as its by-product. For this, we construct a birational morphism (Theorem 4.1)

$$\rho : \mathbf{K} \to \mathbf{N}$$

and then show that this coincides with the morphism $f_\sigma : \mathbf{K} \to \mathbf{K}_\sigma$ of [KL04] by examining the fibers of ρ (Proposition 7.2). M.S. Narasimhan and S. Ramanan conjectured that the desingularization $\mathbf{N}$ can be blown down along certain projective

1991 *Mathematics Subject Classification.* 14H60, 14F25, 14F42.

Key words and phrases. Moduli space, vector bundle, Hecke cycle, desingularization.

Young-Hoon Kiem was partially supported by KOSEF.

fibrations to obtain another nonsingular model of M_0 ([NR78], page 292) and this was proved by N. Nitsure [Nit89]. Our result shows that this blown down process corresponds to the morphism

$$f_\epsilon : \ \mathbf{K}_\sigma(\cong \mathbf{N}) \longrightarrow \mathbf{K}_\epsilon(\cong \mathbf{S}).$$

In summary, the three desingularizations are related by morphisms

$$\mathbf{K} \to \mathbf{N} \to \mathbf{S}$$

which can be described explicitly as blow-up maps along smooth subvarieties.

The strategy of the construction of ρ is similar as that of f in [KL04]. There is a birational map $\rho' : \mathbf{K} \dashrightarrow \mathbf{N}$ which is defined on the open subset M_0^s of stable bundles. By GAGA and Riemann's extension theorem [Mum76], it suffices to show that ρ' can be extended to a continuous map with respect to the usual complex topology. By Luna's slice theorem, for each point $x \in M_0 \backslash M_0^s$, there is an analytic submanifold W of the Quot scheme whose quotient by the stabilizer H of a point in both W and the closed orbit represented by x is analytically equivalent to a neighborhood of x in M_0. Furthermore, Kirwan's desingularization $\tilde{W}/\!\!/H$ of $W/\!\!/H$ is a neighborhood of the preimage of x in $\mathbf{K}$.

There is a universal family $\mathcal{U}$ of rank 2 vector bundles over X parameterized by $\tilde{W}$, which is induced from the universal bundle over the Quot scheme. By applying an elementary modification with respect to the points of the curve X, we have a family $\mathcal{U}'$ of rank 2 vector bundles of determinant $\mathcal{O}_X(-x)$ for some $x \in X$, which is parameterized by the projective bundle $\mathbb{P}\mathcal{U}^*$ over $\tilde{W} \times X$. For any point $w \in \tilde{W}$ lying over a stable bundle in M_0, the bundles of $\mathcal{U}'$ parameterized by the fiber of w are all stable, and a good Hecke cycle is associated to w. This process yields the birational map $\rho' : \mathbf{K} \dashrightarrow \mathbf{N}$.

The problem is that for the points $w \in \tilde{W}$ lying over a strictly semistable bundle in M_0, some points of $\mathbb{P}\mathcal{U}^*$ in the fiber of w parameterize unstable bundles in $\mathcal{U}'$. To remedy this, we blow up $\mathbb{P}\mathcal{U}^*$ and then apply an elementary modification of $\mathcal{U}'$ along the exceptional divisors. Local computations of the transition data show that the resulting family $\mathcal{U}''$ yields an analytic extension $\rho : \mathbf{K} \to \mathbf{N}$ of ρ'.

This paper is organized as follows. In section 2, we explain the elementary modification of vector bundles, focusing on its local computations which will be used repeatedly in this paper. In section 3 and section 4, we briefly review the Narasimhan-Ramanan's and Kirwan's desingularizations respectively. In section 5 and section 6, we construct the birational morphism $\rho : \mathbf{K} \to \mathbf{N}$. In section 7, we examine the fibers of ρ and prove that ρ is in fact a blow-up along a smooth subvariety of $\mathbf{N}$. In section 8, we compute the cohomology of $\mathbf{N}$ using the morphism ρ. We remark that N. Nitsure([Nit89]) computed the third cohomology group $H^3(\mathbf{N}, \mathbb{Z})$ of $\mathbf{N}$.

2. Elementary modification

Let X be a smooth projective curve over the complex number field. Let E be a vector bundle over X and E_x the fiber of E at x. For simplicity, assume $\mathrm{rk}(E) = 2$.

For any nonzero homomorphism $\nu : E_x \to \mathbb{C}$, we have an exact sequence

$$0 \to E^\nu \to E \xrightarrow{\nu} \mathbb{C}_x \to 0, \tag{2.1}$$

where $\mathbb{C}_x$ is the skyscraper sheaf supported at x. Then $E^\nu = \ker(\nu)$ is locally free and is called an *elementary modification* of E.

In terms of the transition matrices, this process can be described as follows. Choose a local trivialization of E with an open covering $\{V_i\}$ of X and the transition matrices

$$\{\ g_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} : V_i \cap V_j \ \longrightarrow\ GL(2, \mathbb{C})\ \}. \tag{2.2}$$

We can refine the covering so that x is contained in V_1 only. Let ζ be a coordinate function on V_1 such that $\zeta(x) = 0$.

Suppose that $\nu : E_x \cong \mathbb{C}^2 \to \mathbb{C}$ is the first projection. Then a local section (f, g) of the sheaf E^ν on V_1 is $(\zeta f, g)$ when considered as a local section of E on V_1. Hence from the computation

$$\begin{pmatrix} f \\ g \end{pmatrix} \leftrightarrow \begin{pmatrix} \zeta f \\ g \end{pmatrix} \mapsto g_{1j} \begin{pmatrix} \zeta f \\ g \end{pmatrix} = \begin{pmatrix} \zeta a_{1j} f + b_{1j} g \\ \zeta c_{1j} f + d_{1j} g \end{pmatrix}, \tag{2.3}$$

the transition matrix of E^ν from V_1 to V_j for $j \neq 1$ is

$$\begin{pmatrix} \zeta a_{1j} & b_{1j} \\ \zeta c_{1j} & d_{1j} \end{pmatrix}.$$

Also, the transition of E^ν from V_j to V_1 for $j \neq 1$ is the inverse matrix

$$\begin{pmatrix} \zeta^{-1} a_{j1} & \zeta^{-1} b_{j1} \\ c_{j1} & d_{j1} \end{pmatrix}$$

and the other transition matrices are unchanged. Note that $E^\nu \cong E^{\lambda\nu}$ for any nonzero $\lambda \in \mathbb{C}$.

In this way, we can produce vector bundles E^ν of determinant $L(-x)$ from E of determinant L. In [NR78], Narasimhan and Ramanan used this process to construct the Hecke cycles, as will be reviewed in next section.

Later we will also use the elementary modification to construct a morphism $\rho : \mathbf{K} \to \mathbf{N}$. It requires the following generalization to higher dimensions. Let S be a smooth complex manifold and let Z be a smooth hypersurface of S. Let E (resp. F) be a vector bundle on S (resp. Z) with $\mathrm{rk}(F) < \mathrm{rk}(E)$. Assume that there is a surjective homomorphism $\nu : E|_Z \to F$. Then the kernel E^ν of the composition $E \to E|_Z \xrightarrow{\nu} F$ is locally free and defines a vector bundles on S. This situation can be summarized in the following diagram(see [Mar87]).

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \ker(\nu) \arrow[r] \arrow[u] & E|_Z \arrow[r, "\nu"] \arrow[u] & F \arrow[r] & 0 \\
0 \arrow[r] & E^\nu \arrow[r] \arrow[u] & E \arrow[r] \arrow[u] & F \arrow[r] \arrow[u, equal] & 0 \\
 & E(-Z) \arrow[r, equal] \arrow[u] & E(-Z) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \tag{2.4}$$

Now let X be an algebraic curve as before, S a complex manifold, and $E \to S\times X$ a family of vector bundles over X parameterized by S. For simplicity, assume $\dim S = 2$. Let $\pi : \tilde{S} \to S$ be the blow-up at one point $\theta \in S$ with the exceptional divisor Z. Suppose that $E_\theta \cong L_1 \oplus L_2$ for some line bundles L_1 and L_2 on X. Let $\tilde{E} := (\pi \times 1_X)^*E$ and $\tilde{L}_i := (\pi \times 1_X)^* L_i$ $(i = 1, 2)$ be the families of bundles parameterized by $\tilde{S}$ and Z respectively so that $\tilde{E}|_{Z\times X} \cong \tilde{L}_1 \oplus \tilde{L}_2$. Consider $\tilde{E}$ over $\tilde{S} \times X$(resp. $\tilde{L}$ over $Z \times X$) as playing the role of E over S(resp. F over Z) in the above. Then we have

Lemma 2.1. *Let ν be the first (resp. second) projection $\tilde{E}|_{Z\times X} \to \tilde{L}_1$. Then the associated elementary modification $\tilde{E}^\nu$ defines a family of vector bundles over X such that for each $\tilde{\theta} \in Z$, $\tilde{E}^\nu|_{\tilde{\theta}\times X}$ is an extension of L_2 by L_1 (resp. L_1 by L_2).*

Proof. Choose a local coordinate (z, t) of S in a small neighborhood U of $\theta = (0, 0)$. Let $\tilde{\theta} \in Z$ represent the line $l_\tau : t = \tau z$ in U for some $\tau \in \mathbb{C}$. Choose an open covering $\{V_i\}$ of X such that $E|_{l_\tau \times V_i}$ are all trivial. Fix a trivialization for each V_i and let $L^\tau_k = \tilde{L}_k|_{l_\tau\times X}$ for $k = 1, 2$. Since $E|_{0\times X} \cong L_1 \oplus L_2$, the transition matrices of $\tilde{E}|_{l_\tau\times X}$ are of the form

$$\begin{pmatrix} \lambda_{ij} & zb_{ij} \\ zc_{ij} & \mu_{ij} \end{pmatrix} \tag{2.5}$$

where $\{\lambda_{ij}|_{z=0}\}$ and $\{\mu_{ij}|_{z=0}\}$ are the transition functions of L_1 and L_2 respectively. From the properties of transition maps, we have

$$(\lambda_{jk}\mu^{-1}_{jk})(\mu^{-1}_{ij} b_{ij}) + (\mu^{-1}_{jk} b_{jk}) = (\mu^{-1}_{ki} b_{ki}).$$

This shows that the data $\{\mu^{-1}_{ij} b_{ij}|_{z=0}\}$ define a Cěch cocycle in $H^1(X, L_1 \otimes L_2^{-1})$. Similarly, the data $\{\lambda^{-1}_{ij} c_{ij}|_{z=0}\}$ define a Cěch cocycle in $H^1(X, L_1^{-1} \otimes L_2)$.

The modified bundle $\tilde{E}^\nu$ over $l_\tau \times X$ is given by the kernel of the composition

$$\tilde{E}|_{l_\tau\times X} \cong L^\tau_1 \oplus L^\tau_2 \to L^\tau_1.$$

Note that any section of $\tilde{E}^\nu$ over $l_\tau \times V_i$ is of the form (zf, g) when considered as a section of $\tilde{E}$. From the computation

$$\begin{pmatrix} f \\ g \end{pmatrix} \leftrightarrow \begin{pmatrix} zf \\ g \end{pmatrix} \mapsto \begin{pmatrix} \lambda_{ij} & zb_{ij} \\ zc_{ij} & \mu_{ij} \end{pmatrix}\begin{pmatrix} zf \\ g \end{pmatrix} = \begin{pmatrix} z(\lambda_{ij} f + b_{ij} g) \\ z^2 c_{ij} f + \mu_{ij} g \end{pmatrix} \leftrightarrow \begin{pmatrix} \lambda_{ij} & b_{ij} \\ z^2c_{ij} & \mu_{ij} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$$

the transition for $\tilde{E}^\nu|_{\tilde{\theta}\times X}$ is

$$\begin{pmatrix} \lambda_{ij} & b_{ij} \\ 0 & \mu_{ij} \end{pmatrix}.$$

Hence $\tilde{E}^\nu|_{\tilde{\theta}\times X}$ is an extension of L_2 by L_1 whose extension class is given by $\{\mu^{-1}_{ij} b_{ij}|_{z=0}\}$. The same argument proves the case of the second projection. □

3. Moduli space of Hecke cycles

Let X be a smooth projective curve of genus $g \geq 3$ over the complex number field. Let M_0 be the moduli space of semistable bundles over X of rank 2 with trivial determinant. Then M_0 is a singular normal projective variety of dimension $3g-3$. Its singular locus is the Kummer variety $\mathfrak{K}$ which consists of the S-equivalence

classes of non-stable bundles $E = L \oplus L^{-1}$ for $L \in Pic^0(X)$. In [NR78], Narasimhan and Ramanan constructed a desingularization

$$\varphi_{\mathbf{N}} : \mathbf{N} \to M_0$$

which is an isomorphism over the open subset M_0^s of stable bundles. The smooth variety $\mathbf{N}$ is called *the moduli space of Hecke cycles.* In this section, we review its construction.

For any point $x \in X$, let M_x be the moduli space of stable vector bundles over X of rank 2 whose determinants are isomorphic to $\mathcal{O}_X(-x)$. Let M_X denote the moduli space of stable bundles over X of rank 2 whose determinants are isomorphic to $\mathcal{O}_X(-x)$ for some $x \in X$, i.e., $M_X = \bigcup_{x \in X} M_x$ inside the moduli space of stable bundles over X of rank 2 and degree 1.

For any stable bundle $E \in M_0^s$ and any $\nu \in \mathbb{P}E_x^*$, we get an associated elementary modification

$$0 \to E^\nu \to E \xrightarrow{\nu} \mathbb{C}_x \to 0$$

so that $\det(E^\nu) = \mathcal{O}_X(-x)$. Since E^ν is again stable([NR78], Lemma 5.5), $E^\nu \in M_x$. Hence by the universal property of M_X, we have a morphism

$$\theta_E : \mathbb{P}E^* \to M_X.$$

More generally, for any family $W \to S \times X$ of stable bundles in M_0^s, there is a canonical morphism $\theta_W : \mathbb{P}W^* \to M_X$. Moreover, this is a closed immersion, provided that $W_{s_1} \ncong W_{s_2}$ whenever $s_1 \neq s_2 \in S$ ([NR78], Lemma 5.9).

From this, we get a morphism

$$\Phi : M_0^s \to Hilb(M_X)$$

into the Hilbert scheme of M_X, defined by $\Phi(E) = \theta_E(\mathbb{P}E^*) \subset M_X$.

Definition 3.1. ([NR78], Definition 5.12) For a stable bundle $E \in M_0^s$, the cycle $\Phi(E)$ in M_X is called the *good Hecke cycle associated to E*. Any subscheme in the irreducible component of $Hilb(M_X)$ containing the good Hecke cycles is called a *Hecke cycle.*

Theorem 3.2. ([NR78], Theorem 5.13) *Via the morphism Φ, M_0^s is isomorphic to an open subscheme of $Hilb(M_X)$ consisting of the good Hecke cycles.* $\square$

To compute the Hilbert polynomial of the good Hecke cycles, we fix an ample line bundle on M_X. Let K_{det} denote the canonical line bundle along the fibers of the fibration $det : M_X \to X$. Then $\mathcal{O}(1) := K_{det}^* \otimes (det)^* K_X$ is an ample line bundle on M_X ([NR78], Lemma 7.1).

Lemma 3.3. ([NR78], Lemma 7.2) *The Hilbert polynomial of a good Hecke cycle is $P(n) = (4n+1)(4n-1)(g-1)$ with respect to $\mathcal{O}(1)$.* $\square$

Recall that the canonical line bundle of M_x is isomorphic to $\mathcal{L}_x^{\otimes(-2)}$ for the ample generator $\mathcal{L}_x$ of $Pic(M_x) \cong \mathbb{Z}$ ([Ram73]). Also, it is known that $\mathcal{L}_x$ is very ample ([BV99]) and we can think M_x as a projective variety embedded in $|\mathcal{L}_x|^*$. In this setting, we see that a good Hecke cycle in M_X restricts to a conic on M_x for each $x \in X$.

Theorem 3.4. ([NR78], §8) *Let* $\mathbf{N}$ *be the irreducible component of* $Hilb^{P(n)}(M_X)$ *containing good Hecke cycles. Then* $\mathbf{N}$ *is a nonsingular variety of dimension* $3g-3$*. Moreover, there is a morphism*

$$\varphi_{\mathbf{N}} : \mathbf{N} \to M_0$$

which is an isomorphism over the set M_0^s *of stable points.* □

The fibers of $\varphi_{\mathbf{N}}$ over the boundary locus $M_0 \backslash M_0^s = \mathfrak{K}$ are described as follows (see [NR78], Proposition 7.8 and Theorem 8.14). First consider $L \in Pic^0(X)$ with $L^2 \ncong \mathcal{O}_X$ and let $l = [L \oplus L^{-1}] \in \mathfrak{K} - \mathfrak{K}_0$ be a non-nodal point in the Kummer variety in M_0. The fiber $\varphi^{-1}(l)$ is isomorphic to the product of two $(g-2)$-dimensional projective spaces $\mathbb{P}H^1(X, L^2)$ and $\mathbb{P}H^1(X, L^{-2})$. Any choice of two points from $\mathbb{P}H^1(X, L^2)$ and $\mathbb{P}H^1(X, L^{-2})$ gives rise to two lines in $\mathbb{P}H^1(X, L^2(-x))$ and $\mathbb{P}H^1(X, L^{-2}(-x))$. It can be shown that these two lines meet at the unique intersection point $\mathbb{P}H^1(X, L^2(-x)) \bigcap \mathbb{P}H^1(X, L^{-2}(-x))$ when we consider their images inside the moduli space M_x. For each $x \in X$, any Hecke cycle in M_X lying over $l \in \mathfrak{K} - \mathfrak{K}_0$ restricts on M_x to this kind of line pairs.

Next consider $L \in Pic^0(X)$ with $L^2 \cong \mathcal{O}_X$ and let $l = [L \oplus L] \in \mathfrak{K}_0$ be a nodal point. The fiber $\varphi^{-1}(l)$ consists of two components $Q_l \cup R_l$: Q_l is the space of all conics which are contained in $\mathbb{P}H^1(X, \mathcal{O})$ and R_l is the space of $\mathcal{O}_{\mathbb{P}^1}(-1)$-thickenings of lines in $\mathbb{P}H^1(X, \mathcal{O})$ which are contained in the thickening of $\mathbb{P}H^1(X, \mathcal{O})_t$ (see [NR78] §3 and §4 for the details). The first variety is isomorphic to a $\mathbb{P}^5$-bundle over $Gr(\mathbb{P}^2, \mathbb{P}^{g-1}) = Gr(3, g)$ of planes in $\mathbb{P}H^1(X, \mathcal{O})$ while the second variety is a $\mathbb{P}^{g-2}$-bundle over the Grassmannian $Gr(\mathbb{P}^1, \mathbb{P}^{g-1}) = Gr(2, g)$ of lines in $\mathbb{P}H^1(X, \mathcal{O})$.

Finally we note that the fine moduli space $\mathbf{N}$ of Hecke cycles in M_X, has the following universal properties.

Proposition 3.5. *(1) Suppose that there is a flat family of closed subschemes of* M_X*,*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, hook] \arrow[dr] & & M_X \times T \arrow[dl] \\ & T & \end{tikzcd}$$

parameterized by T *such that the fiber* $\mathcal{C}_t$ *is a good Hecke cycle for generic* $t \in T$*. Then we have an induced morphism* $\tau : T \to \mathbf{N}$ *such that* $\tau(t) = [\mathcal{C}_t] \in \mathbf{N}$*.*

(2) Suppose a holomorphic map $\tau : T \to \mathbf{N}$ *is given. Suppose* T *is an open subset of a nonsingular quasi-projective variety* W *on which a reductive group* G *acts such that every points in* W *is stable and the smooth geometric quotient* W/G *exists. Furthermore, assume that there is an open dense subset* W' *of* W *such that whenever* $t_1, t_2 \in T \cap W'$ *are in the same orbit, we have* $\tau(t_1) = \tau(t_2)$*. Then* τ *factors through the image* $\bar{T}$ *of* T *in the quotient* W/G*.*

Proof. These are consequences of the universal property of Hilbert scheme and GIT quotients. □

4. Kirwan's desingularization

In this section, we review the Kirwan's desingularization $\mathbf{K}$. Main reference is [Kir86b] and we also refer the reader to [Kie03] for an explicit description of the desingularization process for the case of genus 3 curves.

As we noted before,
$$M_0 = M_0^s \sqcup (\mathfrak{K} - \mathfrak{K}_0) \sqcup \mathfrak{K}_0,$$
where $\mathfrak{K}_0$ consists of the 2^{2g} nodal points in $\mathfrak{K}$. Kirwan's desingularization $\mathbf{K}$ is obtained as a result of systematic blow-ups of M_0. Let M_1 be the blow-up of M_0 along the deepest strata $\mathfrak{K}_0$. By blowing up M_1 along the proper transform of of the middle stratum $\mathfrak{K}$, we get Kirwan's *partial* desingularization M_2. By taking one more blow-up along the singular locus of M_2, we get the *full* desingularization $\mathbf{K}$.

The moduli space M_0 is constructed as the GIT quotient $\mathfrak{R}/\!\!/G$, where $G = SL(p)$ and $\mathfrak{R}$ is a smooth quasi-projective variety which is a subset of the space of holomorphic maps from X to the Grassmannian $Gr(2,p)$ of 2-dimensional quotients of $\mathbb{C}^p$ where p is a large even number.

Let $l \in \mathfrak{K}_0$ represent $L \oplus L^{-1}$ where $L^2 \cong \mathcal{O}_X$. There is a unique closed orbit in $\mathfrak{R}^{ss}$ lying over h. By deformation theory, the normal space of this orbit is
$$H^1(End_0(L \oplus L^{-1})) \cong H^1(\mathcal{O}) \otimes sl(2)$$
where the subscript 0 denotes the trace-free part. By Luna's slice theorem, there is a neighborhood of l homeomorphic to $(H^1(\mathcal{O}) \otimes sl(2))/\!\!/SL(2)$ since the stabilizer of h is $SL(2)$ ([Kir86b], (3.3)). More precisely, there is an $SL(2)$-invariant locally closed subvariety W in $\mathfrak{R}^{ss}$ containing l and an $SL(2)$-invariant morphism $W \to H^1(\mathcal{O}) \otimes sl(2)$, étale at h, such that we have the following commutative diagram with all horizontal morphisms being étale.

$$\begin{array}{ccccc}
G \times_{SL(2)} (H^1(\mathcal{O}) \otimes sl(2)) & \longleftarrow & G \times_{SL(2)} W & \longrightarrow & \mathfrak{R}^{ss} \\
\downarrow & & \downarrow & & \downarrow \\
(H^1(\mathcal{O} \otimes sl(2))/\!\!/SL(2) & \longleftarrow & W/\!\!/SL(2) & \longrightarrow & M_0
\end{array} \tag{4.1}$$

Next, let $l \in \mathfrak{K} - \mathfrak{K}_0$ represent $L \oplus L^{-1}$ with $L^2 \ncong \mathcal{O}$. The normal space to the unique closed orbit over l is isomorphic to
$$H^1(End_0(L \oplus L^{-1})) \cong H^1(\mathcal{O}) \oplus H^1(L^2) \oplus H^1(L^{-2}).$$
Here the stabilizer $\mathbb{C}^*$ acts with weights $0, 2, -2$ respectively on the components, and there is a neighborhood of l homeomorphic to
$$H^1(\mathcal{O}) \bigoplus (H^1(L^2) \oplus H^1(L^{-2})/\!\!/\mathbb{C}^*).$$
Notice that $H^1(\mathcal{O})$ is the tangent space to $\mathfrak{K}$ and hence
$$H^1(L^2) \oplus H^1(L^{-2})/\!\!/\mathbb{C}^* \cong \mathbb{C}^{2g-2}/\!\!/\mathbb{C}^*$$
is the normal cone. The GIT quotient of the projectivization $\mathbb{P}\mathbb{C}^{2g-2}$ by the induced $\mathbb{C}^*$-action is $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$ and the normal cone $\mathbb{C}^{2g-2}/\!\!/\mathbb{C}^*$ is obtained by collapsing the zero section of the line bundle $\mathcal{O}_{\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}}(-1,-1)$.

Let $Z^{ss}_{SL(2)}$ (resp. $Z^{ss}_{\mathbb{C}^*}$) be the set of semiatable points in $\mathfrak{R}^{ss}$ fixed by $SL(2)$ (resp. $\mathbb{C}^*$). Let $\mathfrak{R}_1$ be the blow-up of $\mathfrak{R}^{ss}$ along the smooth subvariety $GZ^{ss}_{SL(2)}$. Then by [Kir85] Lemma 3.11, the GIT quotient $\mathfrak{R}_1^{ss}/\!\!/G$ is the first blow-up M_1 of M_0 along $GZ^{ss}_{SL(2)}/\!\!/G \cong \mathfrak{K}_0$. The $\mathbb{C}^*$-fixed point set in $\mathfrak{R}_1^{ss}$ is the proper transform $\tilde{Z}^{ss}_{\mathbb{C}^*}$ of $Z^{ss}_{\mathbb{C}^*}$ and the quotient of $G\tilde{Z}^{ss}_{\mathbb{C}^*}$ by G is the blow-up $\tilde{\mathfrak{K}}$ of $\mathfrak{K}$ along $\mathfrak{K}_0$. Let $\mathfrak{R}_2$ be the blow-up of $\mathfrak{R}_1^{ss}$ along the smooth subvariety $G\tilde{Z}^{ss}_{\mathbb{C}^*} = G \times_{N^{\mathbb{C}^*}} \tilde{Z}^{ss}_{\mathbb{C}^*}$, where $N^{\mathbb{C}^*}$ is the normalizer of $\mathbb{C}^*$. Then again by [Kir85] Lemma 3.11, the GIT quotient

$\mathfrak{R}_2^{ss}/\!\!/G$ is the second blow-up M_2 of M_1 along $G\tilde{Z}_{\mathbb{C}^*}^{ss}/\!\!/G \cong \tilde{\mathfrak{K}}$. This is Kirwan's partial desingularization of M_0 (see §3 of [Kir86b]).

The points with stabilizer greater than the center $\{\pm 1\}$ in $\mathfrak{R}_2^{ss}$ is precisely the exceptional divisor of the second blow-up and the proper transform $\tilde{\Delta}$ of the subset Δ of the exceptional divisor of the first blow-up, which corresponds, via Luna's slice theorem, to

$$SL(2)\cdot\mathbb{P}\{\begin{pmatrix}0 & b\\ c & 0\end{pmatrix}|b,c\in H^1(\mathcal{O})\} \ \subset\ \mathbb{P}(H^1(\mathcal{O})\otimes sl(2)).$$

Hence by blowing up M_2 along $\tilde{\Delta}/\!\!/SL(2)$, we get a smooth variety $\mathbf{K}$, Kirwan's desingularization.

We can now state the main result of this paper. Note that both $\mathbf{N}$ and $\mathbf{K}$ contain M_0^s as dense open subsets. Hence we have a birational map $\rho' : \mathbf{K} \dashrightarrow \mathbf{N}$.

Theorem 4.1. *ρ' extends to a morphism $\rho : \mathbf{K} \to \mathbf{N}$.*

In the subsequent two sections, we prove this theorem and in section 7 we show that ρ is in fact a blow-up along a smooth subvariety of $\mathbf{N}$. Finally, in section 8 we compute the cohomology of $\mathbf{N}$.

5. Middle stratum

Let us first extend ρ' to points over the middle stratum of M_0. Let $l = [L\oplus L^{-1}] \in \mathfrak{K} - \mathfrak{K}_0$ be a non-nodal point in the Kummer variety and let W be the étale slice of the unique closed orbit in $\mathfrak{R}^{ss}$ over l. The deformation space of $L\oplus L^{-1}$ with determinant fixed is

$$\mathcal{N} = H^1(\mathrm{End}_0(L\oplus L^{-1})) = H^1(\mathcal{O})\oplus H^1(L^2)\oplus H^1(L^{-2}) \tag{5.1}$$

where the subscript 0 above denotes the trace-free part. There is a versal deformation $\mathcal{F}$ over $\mathcal{N}\times X$ and this gives us an analytic isomorphism of a neighborhood U of 0 in $\mathcal{N}$ with a neighborhood of l in W. The restriction of $\mathcal{F}$ to $H^1(\mathcal{O})$ is a direct sum $\mathcal{L}\oplus\mathcal{L}^{-1}$ where $\mathcal{L}$ is the versal deformation of the line bundle L. The group $\mathbb{C}^*$ acts with weights $0, 2, -2$ respectively on the three components of $\mathcal{N}$ in (5.1).

Let $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ be the blow-up along $H^1(\mathcal{O})$ and let $\tilde{\mathcal{N}}^s$ be the set of stable points in $\tilde{\mathcal{N}}$ with respect to the obvious induced action of $\mathbb{C}^*$. Let D be the set of stable points in the exceptional divisor of the blow-up; let $\tilde{\mathcal{F}}$ be the pull-back of $\mathcal{F}$ to $\tilde{\mathcal{N}}^s\times X$; let $\tilde{\mathcal{L}}$ be the pull-back of $\mathcal{L}$ to D; let $\psi : \mathbb{P}\tilde{\mathcal{F}}^* \to \tilde{\mathcal{N}}^s\times X$ be the projectivization of the dual of $\tilde{\mathcal{F}}$. Consider the composition

$$\mathbb{P}\tilde{\mathcal{F}}^*\times X \xrightarrow{\psi\times 1_X} (\tilde{\mathcal{N}}^s\times X)\times X \xrightarrow{p_{13}} \tilde{\mathcal{N}}^s\times X$$

where p_{13} denotes the projection onto the product of the first and the third components. Let $\tilde{\mathcal{F}}'$ be the pull-back of $\tilde{\mathcal{F}}$ via the above composition; let q_X (resp. q_N) be the composition of ψ with the projection onto X (resp. $\tilde{\mathcal{N}}^s$); let $i : \mathbb{P}\tilde{\mathcal{F}}^* \to \mathbb{P}\tilde{\mathcal{F}}^*\times X$ be the map $1_{\mathbb{P}\tilde{\mathcal{F}}^*}\times q_X$. Then there is a tautological homomorphism $\tilde{\mathcal{F}}' \to i_*\mathcal{O}_{\mathbb{P}\tilde{\mathcal{F}}^*}(1)$. Let $\mathcal{E}$ be its kernel. Then $\mathcal{E}$ is a family of rank 2 bundles on X of degree -1 parameterized by $\mathbb{P}\tilde{\mathcal{F}}^*$ since for each $\theta\in\mathbb{P}\tilde{\mathcal{F}}^*$,

$$\mathcal{E}|_{\{\theta\}\times X} = \ker(\tilde{\mathcal{F}}|_{\{q_N(\theta)\}\times X} \twoheadrightarrow \mathcal{O}_{q_X(\theta)}). \tag{5.2}$$

The isomorphism $\tilde{\mathcal{F}}|_{D\times X} \cong \tilde{\mathcal{L}}\oplus\tilde{\mathcal{L}}^{-1}$ gives rise to two sections

$$s_1, s_2 : D\times X \to \mathbb{P}\tilde{\mathcal{F}}^*|_{D\times X}$$

by considering the obvious surjections $\tilde{\mathcal{F}}|_{D\times X} \to \tilde{\mathcal{L}}$ and $\tilde{\mathcal{F}}|_{D\times X} \to \tilde{\mathcal{L}}^{-1}$. Thus we have two disjoint codimension 2 subvarieties $s_1(D\times X)$ and $s_2(D\times X)$ of $\mathbb{P}\tilde{\mathcal{F}}^*$.

Lemma 5.1. *$\mathcal{E}|_{\{\theta\}\times X}$ is stable if and only if $\theta \in \mathbb{P}\tilde{\mathcal{F}}^* - s_1(D\times X) - s_2(D\times X)$.*

Proof. If $q_N(\theta) \notin D$, $\mathcal{F}|_{\{q_N(\theta)\}\times X}$ is a stable bundle and hence $\mathcal{E}|_{\{\theta\}\times X}$ is stable since it is the result of an elementary modification at one point ([NR78] Lemma 5.5). For $\theta \in q_N^{-1}(D) - s_1(D\times X) - s_2(D\times X)$, $\mathcal{E}|_{\{\theta\}\times X}$ is the result of an elementary modification $L_\theta \oplus L_\theta^{-1} \twoheadrightarrow \mathbb{C}$ for $L_\theta = \tilde{\mathcal{L}}|_{\{q_N(\theta)\}\times X}$ where $L_\theta \to L_\theta \oplus L_\theta^{-1} \to \mathbb{C}$ and $L_\theta^{-1} \to L_\theta \oplus L_\theta^{-1} \to \mathbb{C}$ are both nonzero. It is an elementary exercise to show that the result of this modification is a stable bundle, whose isomorphism class is independent of the choice of the map $L_\theta \oplus L_\theta^{-1} \twoheadrightarrow \mathbb{C}$.

If $\theta \in s_1(D\times X)$, $\mathcal{E}|_{\{\theta\}\times X}$ is $L_\theta(-x) \oplus L_\theta^{-1}$ where $x = q_X(\theta)$. Hence it is unstable. Similarly $\mathcal{E}|_{\{\theta\}\times X}$ is unstable for $\theta \in s_2(D\times X)$. □

From the definition (5.2), we have

$$\mathcal{E}|_{s_1(D\times X)\times X} \cong \tilde{\mathcal{L}}^{-1} \oplus \tilde{\mathcal{L}}(-\mathbf{q}_X)$$

where $\tilde{\mathcal{L}}(-\mathbf{q}_X)$ is the kernel of $\tilde{\mathcal{L}} \to \tilde{\mathcal{L}}|_{i(s_1(D\times X))}$. Similarly

$$\mathcal{E}|_{s_2(D\times X)\times X} \cong \tilde{\mathcal{L}} \oplus \tilde{\mathcal{L}}^{-1}(-\mathbf{q}_X).$$

In order to get a family of stable bundles, we blow up $\mathbb{P}\tilde{\mathcal{F}}^*$ along the locus of unstable bundles $s_1(D\times X) \cup s_2(D\times X)$. Let $p: Z \to \mathbb{P}\tilde{\mathcal{F}}^*$ be this blow-up and D', D'' be the exceptional divisors for s_1 and s_2 respectively. Let $\mathcal{E}'$ denote the pull-back of $\mathcal{E}$ to $Z\times X$ and $\mathcal{L}'$ and $\mathcal{L}'(-\mathbf{q}_X)$ (resp. $\mathcal{L}''$ and $\mathcal{L}''(-\mathbf{q}_X)$) be the pull-backs of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}(-\mathbf{q}_X)$ to $D'\times X$ (resp. $D''\times X$). Then we have $\mathcal{E}'|_{D'\times X} \cong \mathcal{L}'^{-1} \oplus \mathcal{L}'(-\mathbf{q}_X)$ and $\mathcal{E}'|_{D''\times X} \cong \mathcal{L}'' \oplus \mathcal{L}''^{-1}(-\mathbf{q}_X)$. Now let

$$\mathbf{E} = \ker\left[\mathcal{E}' \to \mathcal{E}'|_{(D'\cup D'')\times X} \to \mathcal{L}'(-\mathbf{q}_X) \oplus \mathcal{L}''^{-1}(-\mathbf{q}_X)\right].$$

Lemma 5.2. *$\mathbf{E}$ is a family of stable vector bundles of degree -1 on X parameterized by Z.*

Proof. Let θ be any point in $s_1(D\times X)$ and $x = q_X(\theta)$. Let C be a line in $\mathcal{N}$ given by a map $\mathbb{C} \to \mathcal{N}$, $z \mapsto (a, zb, zc)$ for $a \in H^1(\mathcal{O})$, $0 \neq b \in H^1(L^2)$, $0 \neq c \in H^2(L^{-2})$. Note that any point in D is represented by such a line. We consider such a line for $q_N(\theta)$. By restricting to a neighborhood U of 0 in $\mathbb{C}$, we can find a finite open covering $\{V_i\}$ of X such that $\mathcal{F}|_{U\times V_i}$ is trivial and x is contained only in V_1. Fix a trivialization for each i. Then the transition matrix of $F^z := \mathcal{F}|_{\{(a,zb,zc)\}\times X}$ from V_i to V_j is of the form

$$\begin{pmatrix} \lambda_{ij} & zb_{ij} \\ zc_{ij} & \lambda_{ij}^{-1} \end{pmatrix} \tag{5.3}$$

where $\{\lambda_{ij}|_{z=0}\}$ is the transition for $L_a := \mathcal{L}|_{\{(a,0,0)\}\times X}$. Further, b and c are the cohomology classes of the cocycles $\{\lambda_{ij}b_{ij}|_{z=0}\}$ and $\{\lambda_{ij}^{-1}c_{ij}|_{z=0}\}$ in $H^1(L_a^2) \cong H^1(L^2)$ and $H^1(L_a^{-2}) \cong H^1(L^{-2})$ respectively.

The normal space to $s_1(D\times X)$ at θ is a two dimensional space $\{(z,t)\}$ where (z,t) represents the bundle F^z and the surjection $F^z|_x \cong \mathbb{C}^2 \twoheadrightarrow \mathbb{C}$ given by $(1,t)$. By definition, the bundle $\mathcal{E}$ restricted to $(z,t)\times X$ is the kernel of $F^z \to F^z|_x \to \mathbb{C}$. Its transition matrices can be described as follows. Let ζ be a coordinate function

on V_1 such that $\zeta(x)=0$. A section on V_1 of the kernel is of the form $(\zeta f - tg, g)$ for some holomorphic functions f,g. From the computation

(5.4)
$$\begin{pmatrix} f \\ g \end{pmatrix} \leftrightarrow \begin{pmatrix} \zeta f - tg \\ g \end{pmatrix} \mapsto \begin{pmatrix} \lambda_{1j} & zb_{1j} \\ zc_{1j} & \lambda_{1j}^{-1} \end{pmatrix} \begin{pmatrix} \zeta f - tg \\ g \end{pmatrix} = \begin{pmatrix} \zeta\lambda_{1j} & zb_{1j} - t\lambda_{1j} \\ \zeta zc_{1j} & -ztc_{1j} + \lambda_{1j}^{-1} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

the transition matrix from V_1 to V_j for $j \neq 1$ is

$$\begin{pmatrix} \zeta\lambda_{1j} & zb_{1j} - t\lambda_{1j} \\ \zeta zc_{1j} & -ztc_{1j} + \lambda_{1j}^{-1} \end{pmatrix}.$$

The transition from V_j to V_1 for $j \neq 1$ is the inverse matrix

$$\begin{pmatrix} \zeta^{-1}(\lambda_{j1} + ztc_{j1}) & \zeta^{-1}(zb_{j1} + t\lambda_{j1}^{-1}) \\ zc_{j1} & \lambda_{j1}^{-1} \end{pmatrix}$$

and the other transition matrices are unchanged (5.3).

Any point $\tilde\theta$ in Z over θ is represented by a line through 0 in the (z,t)-plane. Suppose $t = \tau z$ for some $\tau \in \mathbb{C}$. When $z=0$ the transition matrices are diagonal and the bundle is just $L_a(-x) \oplus L_a^{-1}$. To get $\mathbf{E}$, we modify $\mathcal{E}$ by the surjection $\mathcal{E}|_{(0,0)} \cong L_a(-x) \oplus L_a^{-1} \to L_a(-x)$. A section over V_1 of $\mathbf{E}$ restricted to the line $t = \tau z$ is of the form (zf, g) for some holomorphic functions f,g. From

$$\begin{aligned} &\begin{pmatrix} f \\ g \end{pmatrix} \leftrightarrow \begin{pmatrix} zf \\ g \end{pmatrix} \mapsto \begin{pmatrix} \zeta\lambda_{1j} & zb_{1j} - \tau z\lambda_{1j} \\ \zeta zc_{1j} & -\tau z^2 c_{1j} + \lambda_{1j}^{-1} \end{pmatrix} \begin{pmatrix} zf \\ g \end{pmatrix} \\ &= \begin{pmatrix} z(\zeta\lambda_{1j} f + (b_{1j} - \tau\lambda_{1j})g) \\ \zeta z^2 c_{1j} f + (-\tau z^2 c_{1j} + \lambda_{1j}^{-1})g \end{pmatrix} \leftrightarrow \begin{pmatrix} \zeta\lambda_{1j} & b_{1j} - \tau\lambda_{1j} \\ \zeta z^2 c_{1j} & -\tau z^2 c_{1j} + \lambda_{1j}^{-1} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \end{aligned} \tag{5.5}$$

we see that the transition matrix of $\mathbf{E}|_{\tilde\theta \times X}$ from V_1 to V_j for $j \neq 1$ is

$$\begin{pmatrix} \zeta\lambda_{1j} & b_{1j} - \tau\lambda_{1j} \\ 0 & \lambda_{1j}^{-1} \end{pmatrix}$$

by plugging in $z=0$. Similarly, the transition from V_j to V_1 for $j \neq 1$ is

$$\begin{pmatrix} \zeta^{-1}\lambda_{j1} & \zeta^{-1}(b_{j1} + \tau\lambda_{j1}^{-1}) \\ 0 & \lambda_{j1}^{-1} \end{pmatrix}$$

and the transition from V_i to V_j for $i \neq 1$, $j \neq 1$ is

$$\begin{pmatrix} \lambda_{ij} & b_{ij} \\ 0 & \lambda_{ij}^{-1} \end{pmatrix}.$$

This implies that $\mathbf{E}|_{\tilde\theta \times X}$ is an extension of L_a^{-1} by $L_a(-x)$. It is an elementary exercise to check that the extension class in $H^1(L_a^2(-x))$ is given by

$$\mu_{ij}^\tau = \begin{cases} \lambda_{1j}(b_{1j} - \tau\lambda_{1j}) & \text{for} \quad i=1, j\neq 1 \\ \zeta^{-1}\lambda_{i1}(b_{i1} + \tau\lambda_{i1}^{-1}) & \text{for} \quad i \neq 1, j = 1 \\ \lambda_{ij} b_{ij} & \text{for} \quad i \neq 1, j \neq 1 \end{cases}$$

Note that

$$\mu_{ij}^0 = \begin{cases} \lambda_{1j} b_{1j} & \text{for} \quad i=1, j\neq 1 \\ \zeta^{-1}\lambda_{i1} b_{i1} & \text{for} \quad i \neq 1, j = 1 \\ \lambda_{ij} b_{ij} & \text{for} \quad i \neq 1, j \neq 1 \end{cases}$$

defines a class in $H^1(L_a^2(-x))$ which is mapped to b via the natural map $H^1(L_a^2(-x)) \to H^1(L_a^2)$. Similarly,

$$\mu_{ij}^{\infty} = \begin{cases} -\lambda_{1j}^2 & \text{for} \quad i = 1, j \neq 1 \\ \zeta^{-1} & \text{for} \quad i \neq 1, j = 1 \\ 0 & \text{for} \quad i \neq 1, j \neq 1 \end{cases}$$

defines a nonzero class in $H^1(L_a^2(-x))$ which generates the kernel of $H^1(L_a^2(-x)) \to H^1(L_a^2)$. Since μ_{ij}^{τ} is a linear combination of μ_{ij}^0 and μ_{ij}^{∞}, the extension classes for $\mathbf{E}|_{p^{-1}(\theta)\times X}$ give us a projective line in $\mathbb{P}H^1(L_a^2(-x))$ which is also the projectivization of the two dimensional subspace given by the preimage of $\mathbb{C}b$. Therefore $\mathbf{E}|_{p^{-1}(\theta)\times X}$ is a family of stable bundles. The same proof shows that $\mathbf{E}|_{\tilde{\theta}\times X}$ is stable for $\tilde{\theta} \in D''$. □

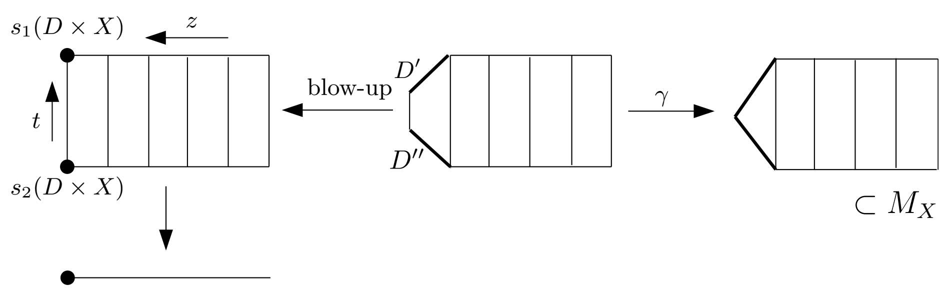

FIGURE 1

As a consequence of the above lemma, we get a morphism $\gamma : Z \to M_X$ over X. By definition $\mathbb{P}\tilde{\mathcal{F}}^*$ is a projective line bundle over $\tilde{\mathcal{N}}^s \times X$ and hence flat over $\tilde{\mathcal{N}}^s \times X$. For $\xi \in D$, $x \in X$, the fiber over $(\xi, x) \in \mathbb{P}\tilde{\mathcal{F}}^*$ in Z is a chain of three rational curves. As remarked in the proof of Lemma 5.1 the isomorphism class of the kernel of $L_a \oplus L_a^{-1} \twoheadrightarrow \mathbb{C}$ is independent of the surjection if neither L_a nor L_a^{-1} is in the kernel. Hence γ is constant on the middle component. The proof of Lemma 5.2 shows that the other two rational curves are embedded by γ into $\mathbb{P}H^1(L_a^2(-x))$ and $\mathbb{P}H^1(L_a^{-2}(-x))$ respectively as projective lines. By [NR78] Proposition 7.8, this implies that the image of $p^{-1}(q_N^{-1}(\xi))$ by γ is a limit Hecke cycle. On the other hand, for $\xi \in \tilde{\mathcal{N}}^s - D$, the image of $p^{-1}(q_N^{-1}(\xi))$ by γ is a good Hecke cycle (Definition 3.1) and thus the Hilbert polynomials of the fibers of the image by γ of Z over $\tilde{\mathcal{N}}^s$ is constant. In particular, $\gamma(Z)$ is a flat family of Hecke cycles in M_X parameterized by $\tilde{\mathcal{N}}^s$. Therefore we proved the following.

Proposition 5.3. *There is an analytic extension $\rho_l : \tilde{\mathcal{N}}^s \to \mathbf{N}$ of the obvious map $\rho_l' : \pi^{-1}(\mathcal{N}^s) \to \mathbf{N}$ which assigns each stable bundle its associated good Hecke cycle where $\mathbf{N}$ is the moduli space of Hecke cycles in M_X.*

Since two isomorphic stable bundles give us the same good Hecke cycles, ρ_l is invariant under the action of $\mathbb{C}^*$ on the open dense subset $\pi^{-1}(\mathcal{N}^s)$ and hence ρ_l is $\mathbb{C}^*$-invariant everywhere. So we get an analytic map

$$\overline{\rho}_l : \tilde{\mathcal{N}}^s /\!\!/ \mathbb{C}^* \to \mathbf{N}.$$

Since a neighborhood of the vertex of the cone $\tilde{\mathcal{N}}^s /\!\!/ \mathbb{C}^*$ is analytically isomorphic to a neighborhood of $l \in \mathfrak{K} - \mathfrak{K}_0 \subset M_0$, we deduce that $\rho : \mathbf{K} \dashrightarrow \mathbf{N}$ extends to the middle stratum analytically.

6. Deepest strata

In this section, we extend ρ' to the points in $\mathbf{K}$ over the deepest strata $\mathfrak{K}_0 = \mathbb{Z}_2^{2g}$. Since the exactly same argument applies to every point in $\mathfrak{K}_0$, we consider only the points in $\mathbf{K}$ over $[\mathcal{O}_X \oplus \mathcal{O}_X] \in M_0$. The deformation space of $\mathcal{O}_X \oplus \mathcal{O}_X$ with determinant fixed is

$$\mathcal{N} = H^1(\mathcal{O}_X) \otimes sl(2)$$

on which $SL(2)$ acts by conjugation on $sl(2)$. There is a versal deformation $\mathcal{F}$ over $\mathcal{N} \times X$ which gives us an analytic isomorphism of a neighborhood of the image $\overline{0}$ of 0 in $\mathcal{N}/\!\!/SL(2)$ with a neighborhood of $[\mathcal{O}_X \oplus \mathcal{O}_X]$ in M_0.

Let Σ be the subset of $\mathcal{N}$ defined by

$$SL(2)\{H^1(\mathcal{O}_X) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$$

which corresponds to the middle stratum of M_0. Let $\pi_1 : \mathcal{N}_1 \to \mathcal{N}$ be the first blow-up in the partial desingularization process, i.e. the blow-up at 0, and let $\mathcal{D}_1^{(1)}$ be the exceptional divisor. Let Δ be the subset of $\mathcal{D}_1^{(1)}$ defined as

$$SL(2)\mathbb{P}\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \in H^1(\mathcal{O}_X)\}$$

and let $\tilde{\Sigma}$ be the proper transform of Σ in $\mathcal{N}_1$. Then the singular locus of $\mathcal{N}_1^{ss}/\!\!/SL(2)$ is the quotient of $\Delta \cup \tilde{\Sigma}$ by $SL(2)$. It is an elementary exercise to check that

$$\mathcal{D}_1^{(1)} \cap \tilde{\Sigma} = SL(2)\mathbb{P}\{H^1(\mathcal{O}_X) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\} = \Delta \cap \tilde{\Sigma}. \tag{6.1}$$

If we remove unstable points that should be deleted after the desingularization process, Δ is the locus in $\mathcal{D}_1^{(1)}$ of 2×2 matrices

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $\dim \operatorname{Span}\{a, b, c\} \leq 2$ while $\Delta \cap \tilde{\Sigma}$ is the locus with $\dim \operatorname{Span}\{a, b, c\} \leq 1$.

Let $\pi_2 : \mathcal{N}_2 \to \mathcal{N}_1$ be the second blow-up, i.e. the blow-up along $\tilde{\Sigma}$ and let $\mathcal{D}_2^{(2)}$ be the exceptional divisor. Let $\mathcal{D}_2^{(1)}$ be the proper transform of $\mathcal{D}_1^{(1)}$. The singular locus of $\mathcal{N}_2/\!\!/SL(2)$ is the quotient of the proper transform $\tilde{\Delta}$ of Δ.

Finally let $\pi_3 : \tilde{\mathcal{N}} = \mathcal{N}_3 \to \mathcal{N}_2$ denote the blow-up of $\mathcal{N}_2$ along $\tilde{\Delta}$ and let $\tilde{\mathcal{D}}^{(3)} = \mathcal{D}_3^{(3)}$ be the exceptional divisor while $\tilde{\mathcal{D}}^{(1)} = \mathcal{D}_3^{(1)}$, $\tilde{\mathcal{D}}^{(2)} = \mathcal{D}_3^{(2)}$ are the proper transforms of $\mathcal{D}_2^{(1)}$ and $\mathcal{D}_2^{(2)}$ respectively. Let $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ be the composition of the three blow-ups. Also let $D_i^{(j)}$ be the quotient of $\mathcal{D}_i^{(j)}$ in $\mathcal{N}_i/\!\!/SL(2)$ for $1 \leq i \leq 3$ and $1 \leq j \leq i$.

6.1. **Modification over $\mathcal{N}_1$.** Let $\mathcal{N}_1^{ss}$ be the set of semistable points in $\mathcal{N}_1$. Let $\mathcal{F}_1$ be the pull-back of $\mathcal{F}$ to $\mathcal{N}_1^{ss} \times X$ and let $\psi_1 : \mathbb{P}\mathcal{F}_1^* \to \mathcal{N}_1^{ss} \times X$ be the projectivization of $\mathcal{F}_1^*$. Consider the composition

$$\mathbb{P}\mathcal{F}_1^* \times X \xrightarrow{\psi_1 \times 1_X} (\mathcal{N}_1^{ss} \times X) \times X \xrightarrow{p_{13}} \mathcal{N}_1^{ss} \times X$$

where p_{13} denotes the projection onto the product of the first and the third components. Let $\mathcal{F}_1'$ be the pull-back of $\mathcal{F}_1$ via the above composition; let q_X (resp. q_N) be the composition of ψ_1 with the projection onto X (resp. $\mathcal{N}_1^{ss}$); let $i : \mathbb{P}\mathcal{F}_1^* \to$

$\mathbb{P}\mathcal{F}_1^* \times X$ be the map $1_{\mathbb{P}\mathcal{F}_1^*} \times q_X$. Then there is a tautological homomorphism $\mathcal{F}_1' \to i_* \mathcal{O}_{\mathbb{P}\mathcal{F}_1^*}(1)$. Let $\mathcal{E}_1$ be its kernel. Then $\mathcal{E}_1$ is a family of rank 2 bundles on X of degree -1 parameterized by $\mathbb{P}\mathcal{F}_1^*$. For $\theta_1 \in q_N^{-1}(\mathcal{D}_1^{(1)})$, $\mathcal{F}_1'|_{\theta_1 \times X} \cong \mathcal{O} \oplus \mathcal{O}$ and $\mathcal{E}_1|_{\theta_1 \times X} \cong \mathcal{O}(-q_X(\theta_1)) \oplus \mathcal{O}$ which is unstable. We modify $\mathcal{E}_1$ to get a family of stable bundles on $\mathcal{D}_1^{(1)} - \tilde{\Sigma}$.

Since $\mathcal{D}_1^{(1)} \subset \pi_1^{-1}(0)$, $\mathcal{F}_1|_{\mathcal{D}_1^{(1)} \times X} \cong \mathcal{O} \oplus \mathcal{O}$ and hence $q_N^{-1}(\mathcal{D}_1^{(1)}) = \mathbb{P}\mathcal{F}_1^*|_{\mathcal{D}_1^{(1)} \times X} = \mathbb{P}^1 \times \mathcal{D}_1^{(1)} \times X$. The restriction $\mathcal{F}_1'|_{q_N^{-1}(\mathcal{D}_1^{(1)}) \times X}$ is thus $\mathcal{O} \oplus \mathcal{O}$ and the tautological homomorphism $\mathcal{F}_1' \to i_* \mathcal{O}_{\mathbb{P}\mathcal{F}_1^*}(1)$ restricted to $q_N^{-1}(\mathcal{D}_1^{(1)}) \times X$ can be factored as

$$\mathcal{F}_1'|_{q_N^{-1}(\mathcal{D}_1^{(1)}) \times X} \cong \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to \mathcal{O}(1)|_{i(q_N^{-1}(\mathcal{D}_1^{(1)}))}$$

where $\mathcal{O}(1)$ denotes the pull-back of $\mathcal{O}_{\mathbb{P}^1}(1)$ by the projection $q_N^{-1}(\mathcal{D}_1^{(1)}) \times X \to \mathbb{P}^1$. Let $\mathcal{O}(-\mathbf{q}_X)$ denote the kernel of the above surjection $\mathcal{O}(1) \to \mathcal{O}(1)|_{i(q_N^{-1}(\mathcal{D}_1^{(1)}))}$ over $q_N^{-1}(\mathcal{D}_1^{(1)}) \times X$. By definition, the composition $\mathcal{E}_1|_{q_N^{-1}(\mathcal{D}_1^{(1)}) \times X} \to \mathcal{F}_1'|_{q_N^{-1}(\mathcal{D}_1^{(1)}) \times X} \to \mathcal{O}(1)|_{i(q_N^{-1}(\mathcal{D}_1^{(1)}))}$ is zero and thus we have a homomorphism

$$\mathcal{E}_1 \to \mathcal{E}_1|_{q_N^{-1}(\mathcal{D}_1^{(1)}) \times X} \to \mathcal{O}(-\mathbf{q}_X).$$

Let $\mathbf{E}_1$ be its kernel.

Lemma 6.1. *Let* $\xi_1 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathcal{D}_1^{(1)} = \mathbb{P}\mathcal{N}$ *with* $a, b, c \in H^1(\mathcal{O}_X)$.

(1) *Suppose* $\dim \operatorname{Span}\{a, b, c\} = 3$. *Then* $\mathbf{E}_1|_{q_N^{-1}(\xi_1) \times X}$ *is a family of stable bundles which gives us a morphism* $\gamma_{\xi_1} : q_N^{-1}(\xi_1) \to M_X$ *over* X. *Furthermore, the image of* $\psi_1^{-1}(\xi_1, x)$ *by* γ_{ξ_1} *for any* $x \in X$ *is a nonsingular conic in* $\mathbb{P}H^1(\mathcal{O}_X) \cong \mathbb{P}H^1(\mathcal{O}_X(-x)) \subset M_x$.

(2) *For* $\xi_1 \in \Delta - \tilde{\Sigma}$, $\mathbf{E}_1|_{q_N^{-1}(\xi_1) \times X}$ *is a family of stable bundles and the map* $\mathbb{P}^1 \cong \psi_1^{-1}(\xi_1, x) \to M_x$ *is a branched double covering onto a projective line in* $\mathbb{P}H^1(\mathcal{O}_X) \cong \mathbb{P}H^1(\mathcal{O}_X(-x)) \subset M_x$.

Proof. We use the same method as in Lemma 5.2. Let $x \in X$ be any. The line $\mathbb{C} \to \mathcal{N}$, $z \mapsto \begin{pmatrix} za & zb \\ zc & -za \end{pmatrix}$ represents ξ_1. By restricting to a neighborhood U of 0 in $\mathbb{C}$, we can find a finite open covering $\{V_i\}$ of X such that $\mathcal{F}|_{U \times V_i}$ is trivial and x is contained only in V_1. Fix a trivialization for each i. The transition matrix of $F^z := \mathcal{F}|_{\{(za,zb,zc)\} \times X}$ from V_i to V_j is of the form

$$\begin{pmatrix} 1 + za_{ij} & zb_{ij} \\ zc_{ij} & 1 - za_{ij} \end{pmatrix} \tag{6.2}$$

mod z^2. Then $\{b_{ij}|_{z=0}\}$ and $\{c_{ij}|_{z=0}\}$ are cocycles represented by $b, c \in H^1(\mathcal{O}_X)$ respectively. The fiber $\mathbb{P}^1$ over (ξ_1, x) in $\mathbb{P}\mathcal{F}_1^*$ has two charts given by

$$(1, t) : \mathbb{C}^2 \to \mathbb{C} \qquad (s, 1) : \mathbb{C}^2 \to \mathbb{C}.$$

Let us consider the first chart $(1, t)$.

By definition, the bundle $\mathcal{E}_1|_{(\xi_1, x, t)}$ is obtained as a consequence of the elementary modification of $F^z|_{z=0}$ at x by $(1, t) : \mathbb{C}^2 \to \mathbb{C}$. Let E_1^z be the kernel of $F^z \to F^z|_x \cong \mathbb{C}^2 \to \mathbb{C}$ where the last map is $(1, t)$ and let ζ be a coordinate function of V_1 with

$\zeta(x) = 0$. The computation (5.4) tells us that the transition matrix of E_1^z from V_1 to V_j for $j \neq 1$ is

$$A_{1j} = \begin{pmatrix} \zeta(1 + za_{1j}) & zb_{1j} - t(1 + za_{1j}) \\ \zeta zc_{1j} & -ztc_{1j} + 1 - za_{1j} \end{pmatrix}.$$

The transition from V_j to V_1 is the inverse matrix $A_{j1} = A_{1j}^{-1}$ and the other transition matrices are unchanged (6.2).

Next $\mathbf{E}_1|_{(\xi_1,x,t)}$ is the result of an elementary modification at $z = 0$ of the family $E_1 = \{E_1^z\} \to \mathbb{C} \times X$ parameterized by $\mathbb{C}$. The transition matrix of E_1^0 from V_1 to V_j for $j \neq 1$ is

$$A_{1j}^0 = \begin{pmatrix} \zeta & -t \\ 0 & 1 \end{pmatrix}$$

Consider the commutative diagram

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "A_{1j}^0"] \arrow[d, "(1{,}0)"'] & \mathbb{C}^2 \arrow[d, "(1{,}t)"] \\ \mathbb{C} \arrow[r, "\zeta"'] & \mathbb{C} \end{tikzcd} \tag{6.3}$$

The horizontal maps are the transitions from V_1 to V_j for E_1^0 and $\mathcal{O}_X(-x)$ respectively. The transitions from V_j to V_1 is the inverse matrices and the other transitions from V_i to V_j $(i, j \neq 1)$ are identity. The vertical maps, which is $(1, 0)$ for V_1 and $(1, t)$ for V_i, $i \neq 1$, give us the surjection $E_1^0 \twoheadrightarrow \mathcal{O}_X(-x)$ and let $\{\mathbf{E}_1^z\}$ be the kernel of

$$E_1 \to E_1|_{z=0} = E_1^0 \twoheadrightarrow \mathcal{O}_X(-x).$$

Then $\mathbf{E}_1^0$ is our $\mathbf{E}_1|_{(\xi_1,x,t)}$. Let us find the transition matrices of $\mathbf{E}_1^0$. From (6.3), a section of the kernel of $E_1 \to \mathcal{O}_X(-x)$ on V_1 is of the form (zf, g) for some holomorphic functions f and g. Also a section of the kernel on V_j is of the form $(zf - tg, g)$. Note that to recover (f, g) from $(zf - tg, g)$ we need to multiply $\begin{pmatrix} z^{-1} & z^{-1}t \\ 0 & 1 \end{pmatrix}$. From the computation

$$\begin{pmatrix} f \\ g \end{pmatrix} \leftrightarrow \begin{pmatrix} zf \\ g \end{pmatrix} \mapsto A_{1j} \begin{pmatrix} zf \\ g \end{pmatrix} \leftrightarrow \begin{pmatrix} z^{-1} & z^{-1}t \\ 0 & 1 \end{pmatrix} A_{1j} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \tag{6.4}$$

we deduce that the transition matrix of $\mathbf{E}_1^z$ from V_1 to V_j $(j \neq 1)$ is

$$\begin{pmatrix} \zeta(1 + za_{1j} + ztc_{1j}) & b_{1j} - 2ta_{1j} - t^2c_{1j} \\ \zeta z^2 c_{1j} & 1 - za_{1j} - ztc_{1j} \end{pmatrix} \tag{6.5}$$

and thus the transition matrix $\mathbf{E}_1^0$ from V_1 to V_j $(j \neq 1)$ is

$$\begin{pmatrix} \zeta & b_{1j} - 2ta_{1j} - t^2c_{1j} \\ 0 & 1 \end{pmatrix}$$

after plugging in $z = 0$. The transition matrix from V_j to V_1 is its inverse

$$\begin{pmatrix} \zeta^{-1} & -\zeta^{-1}(b_{1j} - 2ta_{1j} - t^2c_{1j}) \\ 0 & 1 \end{pmatrix}$$

and the transition from V_i to V_j $(i, j \neq 1)$ is by a similar computation

$$\begin{pmatrix} 1 & b_{ij} - 2ta_{ij} - t^2c_{ij} \\ 0 & 1 \end{pmatrix}$$

This implies that $\mathbf{E}_1|_{(\xi_1,x,t)}$ is an extension of $\mathcal{O}_X$ by $\mathcal{O}_X(-x)$. Via the isomorphism $H^1(\mathcal{O}_X(-x)) \cong H^1(\mathcal{O}_X)$, the extension class is given by

$$\mu_{ij}^t = b_{ij} - 2ta_{ij} - t^2 c_{ij} \tag{6.6}$$

and thus it is $b - 2ta - t^2c$ in $H^1(\mathcal{O}_X)$. If we use $(s,1)$ as our chart on $\mathbb{P}^1$, we get the extension class $s^2b - 2sa - c$ similarly. Therefore, $\mathbf{E}_1|_{\psi_1^{-1}(\xi_1,x)}$ gives us the locus $\{s^2b - 2sta - t^2c \,|\, [s,t] \in \mathbb{P}^1\}$ in $\mathbb{P}H^1(\mathcal{O}_X) \cong \mathbb{P}H^1(\mathcal{O}_X(-x)) \hookrightarrow M_x$. If a, b, c are independent, the locus is a nonsingular conic.

The points in Δ are of the form

$$\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$

after conjugation. In this case, the above locus is a line in $\mathbb{P}H^1(\mathcal{O}_X)$ and the map $\mathbb{P}^1 \cong \psi_1^{-1}(\xi_1, x) \to \mathbb{P}H^1(\mathcal{O}_X)$ is a branched double covering. $\square$

Note that if $\dim \operatorname{Span}\{a,b,c\} = 2$, the matrix $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ is conjugate to matrices of the form

$$\begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$$

The first case becomes unstable in $\mathcal{N}_2$ and thus should be removed after all. The second matrix lies in $\Delta - \tilde{\Sigma}$. Hence the above lemma says $\mathbf{E}_1$ is a family of stable bundles when $\dim \operatorname{Span}\{a,b,c\} \geq 2$. In the next subsection, we deal with the case when $\dim \operatorname{Span}\{a,b,c\} = 1$.

6.2. **Modification over $\mathcal{N}_2$.** Let $\mathcal{F}_2$ be the pull-back of $\mathcal{F}_1$ by $\pi_2 \times 1 : \mathcal{N}_2^s \times X \to \mathcal{N}_1^{ss} \times X$ where $\mathcal{N}_2^s = \mathcal{N}_2^{ss}$ is the set of stable points in $\mathcal{N}_2$. Let $\psi_2 : \mathbb{P}\mathcal{F}_2^* \to \mathcal{N}_2^s \times X$ be the projectivization of $\mathcal{F}_2^*$. By abuse of notation, let q_N (resp. q_X) denote the composition of ψ_2 with the projection onto $\mathcal{N}_2^s$ (resp. X). Then $\mathbb{P}\mathcal{F}_2^*$ is the pull-back of $\mathbb{P}\mathcal{F}_1^*$. Let $\mathbb{P}\mathcal{F}_2^* \to \mathbb{P}\mathcal{F}_1^*$ be the obvious map and let $\mathcal{E}_2$ be the pull-back of $\mathbf{E}_1$ to $\mathbb{P}\mathcal{F}_2^* \times X$.

Lemma 6.2. *The locus of unstable bundles $S = \{\theta \in \mathbb{P}\mathcal{F}_2^* \,|\, \mathcal{E}_2|_{\theta \times X}$ is unstable$\}$ is a smooth subvariety of codimension 2. Furthermore, $\mathcal{E}_2|_{S \times X} \cong \mathcal{L} \oplus \mathcal{M}$ where $\mathcal{L}$ (resp. $\mathcal{M}$) is a family of line bundles of degree 0 (resp. degree -1).*

Proof. The modification of a semistable rank 2 bundle F with $\det F = \mathcal{O}$ on X by $F \to F|_x \cong \mathbb{C}^2 \twoheadrightarrow \mathbb{C}$ is unstable if and only if F is an extension $0 \to L \to F \to L^{-1} \to 0$ for a line bundle L of degree 0 and the surjection $\mathbb{C}^2 \twoheadrightarrow \mathbb{C}$ is $F|_x \to L^{-1}|_x$. For $\xi \in \mathcal{N}_2^s$, $\mathcal{F}_2|_{\xi \times X}$ is a polystable bundle (because non-polystable bundles become unstable in $\mathcal{N}_2$) and the locus of strictly polystable bundles in $\mathcal{N}_1^{ss}$ is $\tilde{\Sigma} \cup \mathcal{D}_1^{(1)}$. Hence S lies over $\mathcal{D}_2^{(1)} \cup \mathcal{D}_2^{(2)}$. But by Lemma 6.1, $\mathbf{E}_1|_{q_N^{-1}(\mathcal{D}_1^{(1)} - \tilde{\Sigma})}$ is a family of stable bundles. Hence in fact S lies over $\mathcal{D}_2^{(2)} = \pi_2^{-1}(\tilde{\Sigma})$.

The proof of Lemma 6.1 says for $\xi_1 = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \in \tilde{\Sigma} \cap \mathcal{D}_1^{(1)}$ and $x \in X$, $\mathbf{E}_1|_{\psi_1^{-1}(\xi_1,x)}$ is a family of extensions of $\mathcal{O}_X$ by $\mathcal{O}_X(-x)$ which splits at exactly two points $(1,0)$ and $(0,1)$. Note that any point in $\tilde{\Sigma} \cap \mathcal{D}_1^{(1)}$ is conjugate to $\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$ for some $a \in H^1(\mathcal{O}_X)$.

For $\xi_1 \in \tilde{\Sigma}$, $\mathcal{F}_1|_{\xi_1\times X}$ is a direct sum of line bundles $L\oplus L^{-1}$ for some line bundle L of degree 0 and the locus of unstable bundles of $\mathbf{E}_1$ in $\psi_1^{-1}(\xi_1,x) = \mathbb{P}\mathcal{F}_1^*|_{(\xi_1,x)} \cong \mathbb{P}^1$ for any $x \in X$ is the two projections $L\oplus L^{-1} \twoheadrightarrow L$ and $L\oplus L^{-1} \twoheadrightarrow L^{-1}$.

Let $J = \{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \,|\, a \in H^1(\mathcal{O}_X)\} \subset \mathcal{N}$. The restriction of $\mathcal{F} \to \mathcal{N}\times X$ to $J\times X$ is $\mathcal{L}\oplus\mathcal{L}^{-1}$ where $\mathcal{L}$ is the versal deformation of the line bundle $\mathcal{O}_X$ over $H^1(\mathcal{O}_X)\times X$ via the isomorphism $J \cong H^1(\mathcal{O}_X)$. Let $\tilde{J}$ be the blow-up of J at 0 and $T \cong \mathbb{C}^*$ be the diagonal torus in $SL(2)$. Then the set of T-fixed points in $\mathcal{N}_1^{ss}$ is $\tilde{J}$ and $\tilde{\Sigma} \cong SL(2)\times_{N^T} \tilde{J}$ where N^T is the normalizer of T in $SL(2)$. (cf.[Kir86b])

Consider the quotient map

$$SL(2)\times\tilde{J} \to SL(2)\times_{N^T}\tilde{J} \cong \tilde{\Sigma}.$$

The pull-back $\mathcal{F}^\dagger$ of $\mathcal{F}_1|_{\tilde{\Sigma}\times X}$ using this map is isomorphic to the pull-back $\mathcal{F}^\sharp$ of $\mathcal{L}\oplus\mathcal{L}^{-1}$ by the projection $SL(2)\times\tilde{J}\to\tilde{J}$. In fact, the isomorphism $\mathcal{F}^\sharp \to \mathcal{F}^\dagger$ over $(g,j)\in SL(2)\times\tilde{J}$ is given by

$$L_j\oplus L_j^{-1} \mapsto g(L_j\oplus L_j^{-1})g^{-1}.$$

Hence the two projections $\mathcal{L}\oplus\mathcal{L}^{-1}\twoheadrightarrow\mathcal{L}$ and $\mathcal{L}\oplus\mathcal{L}^{-1}\twoheadrightarrow\mathcal{L}^{-1}$ in $\mathbb{P}(\mathcal{F}^\sharp)^*$ give us two sections of $\mathbb{P}(\mathcal{F}^\dagger)^*$. Note that if $g\in N^T$, the union of the two sections is mapped to itself by conjugation by g. Since the action of N^T on $SL(2)\times\tilde{J}$ is free, the union of the two sections descends to a smooth subvariety of $\mathbb{P}\mathcal{F}_1^*|_{\tilde{\Sigma}}$. Hence the locus of unstable bundles of $\mathbf{E}_1$ in $\mathbb{P}\mathcal{F}_1^*$ is a codimension 1 smooth subvariety of $\mathbb{P}\mathcal{F}_1^*|_{\tilde{\Sigma}}$. This implies that S is a codimension 2 subvariety of $\mathbb{P}\mathcal{F}_2^*$ lying over $\mathcal{D}_2^{(2)}$.

For the second statement, let $\mathcal{E}^\dagger$ be the kernel of the tautological map from the pull-back of $\mathcal{F}^\dagger$ to $\mathbb{P}(\mathcal{F}^\dagger)^*\times X$ onto $i_*\mathcal{O}_{\mathbb{P}(\mathcal{F}^\dagger)^*}(1)$ where $i:\mathbb{P}(\mathcal{F}^\dagger)^*\to\mathbb{P}(\mathcal{F}^\dagger)^*\times X$ is $1_{\mathbb{P}(\mathcal{F}^\dagger)^*}\times q_X$, exactly as in the construction of $\mathcal{E}_1$ in subsection 6.1. Then it is obvious from the isomorphism $\mathcal{F}^\sharp\cong\mathcal{F}^\dagger$ that $\mathcal{E}^\dagger$ restricted to the two sections is a direct sum of line bundles of degree 0 and -1 respectively. The action of N^T/T interchanges L and L^{-1}. It also interchanges the surjections $L\oplus L^{-1}\twoheadrightarrow L$ and $L\oplus L^{-1}\twoheadrightarrow L^{-1}$. This implies that the line bundles descend to $\tilde{\Sigma}$ and hence we have the desired decomposition of $\mathcal{E}_2|_{S\times X}$. □

To remove unstable bundles from the family $\mathcal{E}_2$ we proceed as in section 5. Let $Z\to\mathbb{P}\mathcal{F}_2^*$ be the blow-up of $\mathbb{P}\mathcal{F}_2^*$ along S; let $\mathcal{D}$ be the exceptional divisor; let $\mathcal{L}'$ (resp. $\mathcal{M}'$) be the pull-back of $\mathcal{L}$ (resp. $\mathcal{M}$) to $\mathcal{D}$; let $\mathcal{E}_2'$ be the pull-back of $\mathcal{E}_2$ to $Z\times X$; let $\mathbf{E}_2$ be the kernel of

$$\mathcal{E}_2'\to\mathcal{E}_2'|_{\mathcal{D}}\cong\mathcal{L}'\oplus\mathcal{M}'\twoheadrightarrow\mathcal{M}'.$$

Lemma 6.3. $\mathbf{E}_2$ *is a family of rank 2 stable bundles of degree* -1.

Proof. For the points over $\tilde{\Sigma}-\Delta$, the proof is identical to Lemma 5.2. For the points over $\tilde{\Sigma}\cap\Delta$, we may assume it lies over $\begin{bmatrix} a & 0 \\ 0 & -a\end{bmatrix}$ for $a\in H^1(\mathcal{O}_X)$ after conjugation. The proof is then identical to that of Lemma 5.2 if we put $\lambda_{ij}=1$. The details are repetition of the same computation and so we omit. □

Consequently, we have a morphism

$$\gamma: Z\to M_X$$

over X. For $\xi \in \mathcal{D}_2^{(2)}$ and $x \in X$, the fiber over (ξ, x) in Z is a chain of 3 rational curves. Since $b = c = 0$ in (6.6), the extension class is a constant multiple of a, and therefore γ is constant on the middle component. As in section 5, each of the other two rational curves is embedded into M_X by γ. When ξ is not in the proper transform $\tilde{\Delta}$ of Δ in $\mathcal{N}_2$, the images of two curves by γ intersect transversely at one point and the image of the fiber (ξ, x) in Z by γ is a limit Hecke cycle as in the middle stratum case. Therefore, we have a family of Hecke cycles in M_X parameterized by $\mathcal{N}_2^s - \tilde{\Delta}$.

If $\xi \in \tilde{\Delta}$, the images of the two rational curves by γ coincide. To get Hecke cycles over $\tilde{\Delta}$ we need to lift the family to $\tilde{\mathcal{N}} = \mathcal{N}_3$.

6.3. **Hecke cycles over $\tilde{\Delta}$.** Recall that $\pi_3 : \tilde{\mathcal{N}} = \mathcal{N}_3 \to \mathcal{N}_2$ is the blow-up of $\mathcal{N}_2$ along $\tilde{\Delta}$, $\tilde{\mathcal{D}}^{(3)} = \mathcal{D}_3^{(3)} = \pi_3^{-1}(\tilde{\Delta})$ and $\tilde{\mathcal{N}}^s$ is the set of stable points in $\mathcal{N}$. Let $\tilde{Z}$ be the pull-back of Z by $\pi_3 \times 1_X$ so that we have the diagram

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\alpha"] \arrow[d, "\tilde{\psi}"'] & Z \arrow[d, "\psi_2"] \\ \tilde{\mathcal{N}}^s \times X \arrow[r, "\pi_3 \times 1_X"] & \mathcal{N}_2^s \times X \end{tikzcd}$$

Let $\tilde{\gamma} : \tilde{Z} \to Z \to M_X$ be the composition of γ with α and consider the diagram

$$\begin{tikzcd} \tilde{Z} \arrow[rr, "\tilde{\gamma} \times q"] \arrow[dr, "q"'] & & M_X \times \tilde{\mathcal{N}}^s \arrow[dl, "p_2"] \\ & \tilde{\mathcal{N}}^s & \end{tikzcd}$$

where $q : \tilde{Z} \to \tilde{\mathcal{N}}^s \times X \to \tilde{\mathcal{N}}^s$ is the composition of $\tilde{\psi}$ with the projection onto $\tilde{\mathcal{N}}^s$ and p_2 is the projection onto the second component. Let Γ be the image of $\tilde{Z}$ by $\tilde{\gamma} \times q$ and ϕ be the restriction of p_2 to Γ. Then Γ is a family of subschemes of M_X parameterized by $\tilde{\mathcal{N}}^s$.

Lemma 6.4. *Γ is a family of Hecke cycles.*

Proof. We have to show that the fiber $\Gamma_\xi := \phi^{-1}(\xi)$ for $\xi \in \tilde{\mathcal{D}}^{(3)}$ is a limit Hecke cycle. Every point in $\tilde{\mathcal{D}}^{(3)}$ represents a normal direction of $\tilde{\Delta}$ in $\mathcal{N}_2^s$. After conjugation, we may assume $\xi_2 := \pi_3(\xi) \in \tilde{\Delta}$ is of the form

$$\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$

for some nonzero $b, c \in H^1(\mathcal{O}_X)$. If we restrict $\mathbf{E}_2$ to the direction normal to $\mathcal{D}_2^{(1)}$ at ξ_2, the transition matrix from V_1 to V_j is given by (6.5) with $a_{1j} = 0$ i.e.

$$\begin{pmatrix} \zeta(1 + tzc_{1j}) & b_{1j} - t^2 c_{1j} \\ \zeta z^2 c_{1j} & 1 - tzc_{1j} \end{pmatrix}$$

mod z^2 and the other transition matrices are given similarly. If ξ represents a direction tangent to $\mathcal{D}_2^{(1)}$ at ξ_2, the transition matrix of $\mathbf{E}_2$ from V_1 to V_j is

$$\begin{pmatrix} \zeta & b_{1j} - t^2 c_{1j} - 2tza_{1j} \\ 0 & 1 \end{pmatrix}$$

by replacing a_{1j} by za_{1j} in (6.5) and the other transition matrices are given similarly. In general, the normal direction represented by ξ is a combination of the above two cases. Hence the transition from V_1 to V_j is

$$\begin{pmatrix} \zeta(1+tzc_{1j}) & b_{1j}-t^2c_{1j}-2tza_{1j} \\ \zeta z^2 c_{1j} & 1-tzc_{1j} \end{pmatrix}$$

and the other transition matrices are given similarly. Thus the first order variation in z for the transition from V_1 to V_j is

$$t\begin{pmatrix} \zeta c_{1j} & -2a_{1j} \\ 0 & -c_{1j} \end{pmatrix} \tag{6.7}$$

and those for the other transitions are given similarly.

The image of $\psi_2^{-1}(\xi_2, x)$ for $x \in X$ is a projective line $\mathbb{P}^1$ in $\mathbb{P}H^1(\mathcal{O}_X(-x))$ from Lemma 6.1 and t is a section of $\mathcal{O}_{\mathbb{P}^1}(1)$. Furthermore, the matrix in (6.7) represents a tangent vector in the moduli space of "triangular bundles" $\mathbb{P}D$ over the Jacobian Jac_0 for X of degree 0, whose fiber over $L \in Jac_0$ is $\mathbb{P}H^1(L^2(-x))$. See [NR78] §6.

Now observe that γ is invariant under the $\mathbb{Z}_2$-action given by $z \to -z$. In fact, the stabilizer of $\xi_2 \in \tilde{\Delta}$ in $SL(2)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$. The first factor $\mathbb{Z}_2$ is the center of $SL(2)$ and acts trivially everywhere. But the second factor $\mathbb{Z}_2$ acts as -1 on the normal directions. Hence the scheme theoretic fiber of Γ over ξ is the projective line thickened by (6.7). This is more precisely the thickening of $\mathbb{P}^1$ by $\mathcal{O}_{\mathbb{P}^1}(-1)$ (since t is a section of $\mathcal{O}_{\mathbb{P}^1}(1)$) inside $\mathbb{P}D$. By [NR78] Proposition 7.8, the fiber Γ_ξ is a limit Hecke cycle. □

By the above lemma, we have a map $\rho_0 : \tilde{\mathcal{N}}^s \to \mathbf{N}$. Since ρ_0 is $SL(2)$-invariant on the dense open subset $\pi^{-1}(\mathcal{N}^s)$, it is invariant everywhere. Therefore, we have a continuous map

$$\overline{\rho}_0 : \tilde{\mathcal{N}}^s/SL(2) \to \mathbf{N}$$

which implies that ρ' extends to everywhere in $\mathbf{K}$.

7. Blowing down Kirwan's desingularization

Based on O'Grady's work [OGr99], it is shown in [KL04] that $\mathbf{K}$ can be blown down twice

$$f: \qquad \mathbf{K} \xrightarrow{f_\sigma} \mathbf{K}_\sigma \xrightarrow{f_\epsilon} \mathbf{K}_\epsilon\ . \tag{7.1}$$

Furthermore, they show in [KL04] that $\mathbf{K}_\epsilon$ is isomorphic to Seshadri's desingularization of M_0 defined in [Ses77]. In this section, we show that the moduli of Hecke cycles $\mathbf{N}$ is in fact the intermediate variety $\mathbf{K}_\sigma$ which was conjectured in [KL04].

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the tautological rank 2 (resp. rank 3) bundle over the Grassmannian $Gr(2,g)$ (resp. $Gr(3,g)$). Let $W = sl(2)^\vee$ be the dual vector space of $sl(2)$. Fix $B \in Gr(3,g)$. Then the variety of complete conics $\mathbf{CC}(B)$ is the blow-up

$$\mathbb{P}(S^2B) \xleftarrow{\Phi_B} \mathbf{CC}(B) \xrightarrow{\Phi_B^\vee} \mathbb{P}(S^2B^\vee)$$

of both of the spaces of conics in $\mathbb{P}B$ and $\mathbb{P}B^\vee$ along the locus of rank 1 conics. We recall the following from [KL04] section 5.

Proposition 7.1. (1) *$\tilde{D}^{(1)}$ is the variety of complete conics $\mathbf{CC}(\mathcal{B})$ over $Gr(3,g)$. In other words, $\tilde{D}^{(1)}$ is the blow-up of the projective bundle $\mathbb{P}(S^2\mathcal{B})$ along the locus of rank 1 conics.*

(2) *There is an integer l such that*

$$\tilde{D}^{(3)} \cong \mathbb{P}(S^2\mathcal{A}) \times_{Gr(2,g)} \mathbb{P}(\mathbb{C}^g/\mathcal{A} \oplus \mathcal{O}(l)).$$

Hence $\tilde{D}^{(3)}$ is a $\mathbb{P}^2 \times \mathbb{P}^{g-2}$ bundle over $Gr(2,g)$.

(3) *The intersection $\tilde{D}^{(1)} \cap \tilde{D}^{(3)}$ is isomorphic to the fibred product*

$$\mathbb{P}(S^2\mathcal{A}) \times \mathbb{P}(\mathbb{C}^g/\mathcal{A})$$

over $Gr(2,g)$. As a subvariety of $\tilde{D}^{(1)}$, $\tilde{D}^{(1)} \cap \tilde{D}^{(3)}$ is the exceptional divisor of the blow-up $\mathbf{CC}(\mathcal{B}) \to \mathbb{P}(S^2\mathcal{B}^\vee)$.

(4) *The intersection $\tilde{D}^{(1)} \cap \tilde{D}^{(2)} \cap \tilde{D}^{(3)}$ is isomorphic to*

$$\mathbb{P}(S^2\mathcal{A})_1 \times \mathbb{P}(\mathbb{C}^g/\mathcal{A})$$

over $Gr(2,g)$ where $\mathbb{P}(S^2\mathcal{A})_1$ denotes the locus of rank 1 quadratic forms.

(5) *The intersection $\tilde{D}^{(1)} \cap \tilde{D}^{(2)}$ is the exceptional divisor of the blow-up $\mathbf{CC}(\mathcal{B}) \to \mathbb{P}(S^2\mathcal{B})$.*

Let σ be the class of lines in the fiber of $\Phi_B^\vee$. Then σ gives us an extremal ray with respect to the canonical bundle of $\mathbf{K}$ and thus we can contract the ray. This turns out to be the contraction of the $\mathbb{P}(S^2\mathcal{A})$-direction of $\tilde{D}^{(3)}$ and the contraction is a blow-down map f_σ. See section 5 of [KL04] for details.

Proposition 7.2. *$\rho : \mathbf{K} \to \mathbf{N}$ factors through $\mathbf{K}_\sigma$ and we have an isomorphism $\mathbf{K}_\sigma \cong \mathbf{N}$.*

Proof. By Riemann's extension theorem [Mum76], it suffices to show that ρ is constant on the fibers of f_σ. From Proposition 7.1, we know f_σ is the result of contracting the fibers $\mathbb{P}^2$ of

$$\tilde{D}^{(3)} = \mathbb{P}(S^2\mathcal{A}) \times \mathbb{P}(\mathbb{C}^g/\mathcal{A} \oplus \mathcal{O}(l)) \to \mathbb{P}(\mathbb{C}^g/\mathcal{A} \oplus \mathcal{O}(l))$$

which amounts to forgetting the choice of b, c in the 2-dimensional subspace of $H^1(\mathcal{O})$ spanned by b, c. From our description of the transition matrices of $\mathbf{E}_2$ in subsection 6.2 and the thickening in subsection 6.3, it is easy to see that the Hecke cycles on $\tilde{\mathcal{D}}^{(3)}$ depends on the two dimensional subspace spanned by $\{b, c\}$ in $H^1(\mathcal{O}_X)$ but not on the choices of b, c in the subspace. Hence ρ factors through $\mathbf{K}_\sigma$.

Now ρ is an isomorphism over the stable part M_0^s of M_0. Further, the divisor $\tilde{D}^{(1)}$ is mapped to the divisor Q_k in [NR78] section 7.7 and the divisor $\tilde{D}^{(2)}$ is mapped to the divisor given by R_k in [NR78] 7.7. The complements of (the images of) these sets in $\mathbf{K}_\sigma$ and $\mathbf{N}$ are of codimension ≥ 2. Now by Zariski's main theorem, we conclude that the induced map from $\mathbf{K}_\sigma$ to $\mathbf{N}$ is an isomorphism. □

Remark 7.3. M.S. Narasimhan and S. Ramanan conjectured that the desingularization $\mathbf{N}$ can be blown down along certain projective fibrations to obtain another nonsingular model of M_0([NR78], page 292) and this was proved by N. Nitsure([Nit89], Proposition 4.A.1 and 4.A.2). Our results, combined with [KL04], show that this blown-down process corresponds to the morphism

$$f_\epsilon : \mathbf{K}_\sigma(\cong \mathbf{N}) \longrightarrow \mathbf{K}_\epsilon(\cong \mathbf{S}).$$

See [KL04] §5 for the structure of the morphism f_ϵ.

8. Cohomology computation

In this section we compute the cohomology of the moduli of Hecke cycles. For a variety T, let

$$P(T) = \sum_{k=0}^{\infty} t^k \dim H^k(T)$$

be the Poincaré series of T. In [Kir85], Kirwan described an algorithm for the Poincaré series of a partial desingularization of a good quotient of a smooth projective variety and in [Kir86b] the algorithm was applied to the moduli space without fixing the determinant. For $P(M_2)$ we use Kirwan's algorithm in [Kir85].

Recall that $M_0 = \mathfrak{R}^{ss}/\!\!/G$ where $G = SL(p)$ and $\mathfrak{R}$ is a subset of the space of holomorphic maps from X to $Gr(2,p)$ for any sufficiently large even integer p ([Kir86b] section 2). By [AB82] §11 and [Kir86a] §13.1, it is well-known that the equivariant Poincaré series $P^G(\mathfrak{R}^{ss}) = \sum_{k\geq 0} t^k \dim H^k_G(\mathfrak{R}^{ss})$ is

$$\frac{(1+t^3)^{2g} - t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} + O(t^k)$$

where k tends to infinity with p. Fix p large enough so that $k > 6g-6$. In order to get $\mathfrak{R}_1^{ss}$ we blow up $\mathfrak{R}^{ss}$ along $GZ^{ss}_{SL(2)}$ and delete the unstable strata. So we get

$$P^G(\mathfrak{R}_1^{ss}) = P^G(\mathfrak{R}^{ss}) + 2^{2g}\Big(\frac{t^2+t^4+\cdots+t^{6g-2}}{1-t^4} - \frac{t^{4g-2}(1+t^2+\cdots+t^{2g-2})}{1-t^2}\Big).$$

Now $\mathfrak{R}_2^{ss}$ is obtained by blowing up $\mathfrak{R}_1^{ss}$ along $G\tilde{Z}^{ss}_{\mathbb{C}^*}$ and deleting the unstable strata. Thus we have

(8.1)
$$\begin{aligned} P^G(\mathfrak{R}_2^{ss}) = P^G(\mathfrak{R}_1^{ss}) \quad & +(t^2+t^4+\cdots+t^{4g-6})\big(\tfrac{1}{2}\tfrac{(1+t)^{2g}}{1-t^2} + \tfrac{1}{2}\tfrac{(1-t)^{2g}}{1+t^2} + 2^{2g}\tfrac{t^2+\cdots+t^{2g-2}}{1-t^4}\big) \\ & -\tfrac{t^{2g-2}(1+t^2+\cdots+t^{2g-4})}{1-t^2}\big((1+t)^{2g} + 2^{2g}(t^2+t^4+\cdots+t^{2g-2})\big). \end{aligned}$$

Because the stabilizers of the G action on $\mathfrak{R}_2^{ss}$ are all finite, we have

$$H^*_G(\mathfrak{R}_2^{ss}) \cong H^*(\mathfrak{R}_2^{ss}/G) = H^*(M_2)$$

and hence we deduce that

$$\begin{aligned} P(M_2) \quad &= \tfrac{(1+t^3)^{2g}-t^{2g+2}(1+t)^{2g}}{(1-t^2)(1-t^4)} \\ &+2^{2g}\big(\tfrac{t^2+t^4+\cdots+t^{6g-2}}{1-t^4} - \tfrac{t^{4g-2}(1+t^2+\cdots+t^{2g-2})}{1-t^2}\big) \\ &+(t^2+t^4+\cdots+t^{4g-6})\big(\tfrac{1}{2}\tfrac{(1+t)^{2g}}{1-t^2} + \tfrac{1}{2}\tfrac{(1-t)^{2g}}{1+t^2} + 2^{2g}\tfrac{t^2+\cdots+t^{2g-2}}{1-t^4}\big) \\ &-\tfrac{t^{2g-2}(1+t^2+\cdots+t^{2g-4})}{1-t^2}\big((1+t)^{2g} + 2^{2g}(t^2+t^4+\cdots+t^{2g-2})\big). \end{aligned} \tag{8.2}$$

Kirwan's desingularization is the blow-up of M_2 along $\tilde{\Delta}/\!\!/SL(2)$ which is isomorphic to the 2^{2g} copies of $\mathbb{P}(S^2\mathcal{A})$ over $Gr(2,g)$. Hence,

$$P(\mathbf{K}) = P(M_2) + 2^{2g}(1+t^2+t^4)P(Gr(2,g))(t^2+t^4+\cdots+t^{2g-4})$$

by [GH78] p. 605.[1]

[1]The formula in [GH78] is stated for smooth manifolds. But the same Mayer-Vietoris argument gives us the same formula in our case (of orbifold M_2 blown up along a smooth subvariety). The only thing to be checked is that the pull-back homomorphism $H^*(M_2) \to H^*(\mathbf{K})$ is injective but this is easy.

On the other hand, $\mathbf{K}$ is the blow-up of $\mathbf{K}_\sigma$ along a $\mathbb{P}^{g-2}$-bundle over $Gr(2,g)$. Hence,

$$\begin{aligned} P(\mathbf{N})=P(\mathbf{K}_\sigma) &= P(\mathbf{K})-2^{2g}(1+t^2+\cdots+t^{2g-4})P(Gr(2,g))(t^2+t^4)\\ &= P(M_2)+2^{2g}P(Gr(2,g))\tfrac{t^6-t^{2g-2}}{1-t^2}. \end{aligned}$$

By Schubert calculus [GH78], we have

$$P(Gr(2,g))=\frac{(1-t^{2g})(1-t^{2g-2})}{(1-t^2)(1-t^4)}$$

and hence we proved the following.

Proposition 8.1. *The Poincaré polynomial of* $\mathbf{N}$ *is*

$$\begin{aligned} P(\mathbf{N}) \quad = \quad & \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)}-\frac{t^{2g-2}(1+t^2+t^4)(1+t)^{2g}}{(1-t^2)(1-t^4)}\\ &+\frac{t^2}{2(1-t^2)}\Big[\frac{(1+t^{4g-6})(1+t)^{2g}}{1-t^2}+\frac{(1-t^{4g-6})(1-t)^{2g}}{1+t^2}\Big]\\ &+2^{2g}\Big[\frac{t^2(1-t^{6g-6})(1+t^4)}{(1-t^2)^2(1-t^4)}-\frac{t^{2g-2}(1-t^6)(1-t^{2g})(1+t^4)}{(1-t^2)^3(1-t^4)}\Big]. \end{aligned}$$

Note that each term in the equality satisfies Poincaré duality i.e. $f(t)=t^{6g-6}f(t^{-1})$.

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